THE PISTON PROBLEM FOR THE EQUATIONS OF SOIL DYNAMICS

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Many authors have studied the one-dimensional plane-wave motion of a medium which has a stress-strain dependence of a complicated type (nonlinear or even irreversible). One may mention here the work of Donnell [1], Taylor [2], von Karman and Duwez [3], Rakhmatulin [4,5,6], Shapiro [7], Barenblatt [8], White and Griffis [9,10], Galin [11], Liakhov and Poliakova [12,13], and others.

Of great interest was the work of Rakhmatulin [4] in which an unloading wave was discovered and studied. Evidently, the work of Barenblatt [8] was the first to establish a clear dependence of the qualitative features of the solution of the piston problem for a nonlinear elastic material on the differential properties of the stress-strain diagram. In particular, a solution with a slow impact wave was constructed which, it developed subsequently, could be used to explain certain qualitative effects that have been observed in the propagation of explosive waves in soil [14-16]. Galin's paper [11] gave a generalization of the results of [8] to the case when stresses depend not only on the strains but on the temperature as well. Most of the work in this direction had in view an application of the results to nonlinear elastic materials and elastic-plastic metals. In these papers it was assumed that the relation between the axial stresses and strains was given in some form or other, and then the mathematical problem arising from this formulation was solved. The question of what system of general three-dimensional equations of motion gave rise to the one-dimensional problem studied was not, however, investigated.

We systematically study below the one-dimensional self-similar problem of soil motion which is excited by the penetration into the soil, or withdrawal out of the soil, of a piston moving at a constant velocity. This study is carried out on the basis of the general system of equations of soil mechanics which is contained in [14,15]. Further, as in [17], the investigation is carried out without explicitly specifying the function which characterizes the material. Therefore, the results which are obtained are of a general nature. As a consequence of this analysis, the following qualitative results are established.

1. The restriction on the form of the function $F(p)$ which was formulated in [17] is necessary here also, though for another reason. It is necessary here for the sound velocity of rarefaction waves of plastic shear to be real.

2. The discontinuous change in the sound velocity with a change in density during the transition from elastic to plastic shear strain, both in rarefaction and compression motion, leads to two possibilities. These depend on the differential properties of the characteristic of the medium. There is the possibility of a doubly-centered wave of compression (or rarefaction) which divides the region of progressive motion, or the possibility of two shock waves of compression (or rarefaction) propagating one after the other at different velocities.*

3. Under specific differential properties of the characteristic of the medium, shock waves of rarefaction may exist.

4. The experimentally established [16] form of the function $F(p)$; $= (kp + b)^2$ turns out to be an optimum in a certain sense. It allows one to reduce the question of the existence of various shock or continuous simple waves to the investigation of the differential properties of a single function $p = f(\rho, \rho_*)$.

5. The presence of a break in the $p = p(\rho)$ diagram in the transition from the $\rho_* =$ const to the branch $\rho = \rho_*$ also leads to the possible existence of additional shock or continuous compression waves.

1. In the case of one-dimensional plane-wave motion, the system of equations describing the motion of the medium has the form [17]

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* We note that in the solutions given in [4], [7], [12] and others, the emergence of centered waves that divide the region of progressive motion is tied in with the approximation of a smooth stress-strain curve by a broken one. That is, it is the consequence of an approximate method of solution of the problem. Here, however, it is a property of an exact solution based on a definite physical mechanism.

$$
\begin{gathered}
\rho\frac{du}{dt}-\frac{\partial\sigma}{\partial x}=0, \qquad \frac{d\rho}{dt}+\rho\frac{\partial u}{\partial x}=0 \\
\frac{d(\sigma+p)}{dt}+\lambda(\sigma+p)=\frac{4}{3}G\frac{\partial u}{\partial x} \\
p=f(\rho,\rho_*)\,e(\rho_*-\rho)\,e(\rho-\rho_0)\equiv f^\circ(\rho,\rho_*) \\
\frac{d\rho_*}{dt}=\frac{d\rho}{dt}e(\rho-\rho_*)\,e\left(\frac{d\rho}{dt}\right) \\
\lambda=\frac{2GW-F'(p)\,dp/dt}{2F(p)}e\,[J_2-F(p)]\,e\left[2GW-F'(p)\frac{dp}{dt}\right] \\
2GW=2G(\sigma+p)\frac{\partial u}{\partial x}, \qquad J_2=\frac{3}{4}(p+\sigma)^2
\end{gathered}
\tag{1.1}
$$

The same notation is used in these formulas as in [17]. The lateral principal stress $\sigma_{yy}=\sigma_1$ is determined by the formula

$$\sigma_1=-p-\tfrac{1}{2}(p+\sigma) \tag{1.2}$$

If the motion is resisted by the compression of an element of the medium, then it is natural to assume that $\sigma<\sigma_1$, that is, by virtue of (1.2) that $\sigma+p<0$. For extension of an element $\sigma+p>0$.

If shear proceeds elastically, then $\lambda=0$ and $J_2<F(p)$. Therefore, under elastic shear we must have the inequalities

$$-\frac{2}{3}\sqrt{3F(p)}<p+\sigma<\frac{2}{3}\sqrt{3F(p)} \tag{1.3}$$

The transition to plastic shear under compression corresponds to the left inequality in (1.3) turning into an equality. Under extension the right inequality becomes an equality.

With the help of the second of relations (1.1), the third may be transformed into the form

$$\frac{d}{dt}\left(\sigma+p+\frac{4}{3}\int\frac{G\,d\rho}{\rho}\right)+\lambda(\sigma+p)=0 \tag{1.4}$$

The piston problem is self-similar because the system of equations for the medium contains no constants whose dimensions differ from those of stress and density. If the velocity of the piston is denoted by V then we have

$$
\begin{gathered}
\sigma=kS(\xi), \qquad p=kP(\xi), \qquad \rho=\rho_1R(\xi), \qquad \rho_*=\rho_1R_*(\xi) \\
u=VU(\xi), \qquad G=G_0g(R), \qquad f^\circ(\rho,\rho_*)=k\,f_*^\circ(R,R_*) \\
F(p)=k^2F(P),\ J_2=k^2I_2, \qquad \xi=x/Vt
\end{gathered}
\tag{1.5}
$$

where ρ_1, G_0 and k are values of density, shear modulus and pressure that characterize the medium (we assume that the modulus G depends on density).

Then the system (1.1), (1.4) goes over to a system of ordinary differential equations for self-similar motion (primes denote differentiation with respect to ξ)

$$R(U-\xi)U' - \frac{1}{m^2}S' = 0, \qquad (U-\xi)R' + RU' = 0$$

$$(U-\xi)\left(S + P + \frac{4}{3}n\int\frac{g(R)\,dR}{R}\right)' + \lambda_*(S+P) = 0$$

$$R_*' = R'e(R-R_*)\,e\left[\frac{1}{t}(U-\xi)R'\right] \tag{1.6}$$

$$\lambda_* \equiv \lambda t, P = f_*^\circ(R, R_*), \qquad I_2 = \frac{3}{4}(S+P)^2$$

$$m^2 = \rho_1\frac{V^2}{k}, \qquad n = \frac{G_0}{k}$$

Relations (1.2) and (1.3) retain their form in the new notation.

If the shear occurs elastically, then $\lambda = 0$, and the third of equations (1.6) goes over to

$$(U-\xi)\left(S+P+\frac{4}{3}n\int\frac{g(R)\,dR}{R}\right)' = 0 \tag{1.7}$$

Under plastic shear $\lambda > 0$ and

$$S + P = \pm\frac{2}{3}\sqrt{3F(P)} \tag{1.8}$$

the upper sign corresponding to rarefaction and the lower to compression.

In this self-similar problem all of the quantities, including R_*, are functions of ξ. In part of the region the specified solution R_* can change with a change in ξ, and then $R_* = R$; in the remaining parts of this region R_* will be constant. Therefore in each of these parts P will be a single-valued function only of R. Hence, in the integration of the system (1.6), (1.7), (1.8) it may be assumed that $P = P(R)$. The dependence of P on R_* and the change in R_* become essential only for a transition through a value of ξ for which the solution either undergoes a discontinuity or for which there occurs a transition from the region where $R_* =$ const to a region where $R_* = R$.

Since $U - \xi \neq 0$ (otherwise the second of equations (1.6) would be violated), we have from (1.7)

$$S = -P - \frac{4}{3}n\int\frac{g(R)\,dR}{R} + \text{const} \tag{1.9}$$

In view of the relative dependence of P on R indicated above, it is clear that both in elastic and plastic shear S is a single-valued function of R ((1.8) and (1.9)). This greatly simplifies the problem, reducing it to the integration of a system of two ordinary equations

$$(U-\xi)R' + RU' = 0, \qquad (U-\xi)RU' + \frac{1}{m^2}a^2R' = 0 \tag{1.10}$$

where

$$a^2 = a_e^2 = -\frac{dS_e}{dR} = P'(R) + \frac{4}{3}n\,\frac{g(R)}{R} \tag{1.11}$$

for elastic shear and

$$a^2 = a_p^2 = -\frac{dS_p}{dR} = P'(R)\left[1 \mp \frac{F'(P)}{\sqrt{3F(P)}}\right] \tag{1.12}$$

for plastic shear.

The quantity a/m plays the role of a sound velocity. From Formulas (1.11) and (1.12) it is seen that for a continuous transition from the region of elastic shear to the region of plastic shear the sound velocity, generally speaking, changes by a jump. We shall see below that interesting peculiarities of the studied motion of the medium are associated with this circumstance.

Further, from (1.12) it is clear that for

$$F'(P) > \sqrt{3F(P)} \tag{1.13}$$

the sound velocity in plastic shear under conditions of rarefaction (upper sign) becomes an imaginary quantity. It is curious to note that this same condition led, in the quasistatic problem with central symmetry that was investigated in [17], to the emergence of limiting lines; therefore, the requirement imposed there that the function $F(P)$ should not have the property (1.13) is also necessary here in the investigation of the dynamic problem. We remark again that the function $F(p)$ which is formulated according to the results of experiments on sandy soil [16] satisfies this requirement.

As is well known, Equations (1.10) have the general solution

$$U = \text{const}, \qquad R = \text{const} \tag{1.14}$$

determining progressive flow, and the particular solution of a simple centered wave

$$U = \xi \pm \frac{1}{m}a, \qquad U = \mp\frac{1}{m}\int\frac{a(R)\,dR}{R} \tag{1.15}$$

The solution of the piston problem, as well as that of the general problem of the decomposition of an arbitrary shock in soil, can be constructed from the solutions (1.14) and (1.15), taking into account the fact that the dependence of a on R is different in the elastic and plastic (in shear) regions ((1.11), (1.12)). For solutions of this type it is necessary to study the relationships on the surface of the shock, which, generally speaking, will certainly arise in the problems studied. The shock surfaces are investigated to some extent below. Now we examine some properties of the solution (1.15).

By the selection of the positive direction of the x-axis, one can arrange that the solution (1.15) will always have the form

$$U = \xi - \frac{1}{m} a, \qquad U = \frac{1}{m} \int_{R_0}^{R} \frac{a(R)\, dR}{R} + U_0 \tag{1.16}$$

Here R_0, U_0 are arbitrary constants. This solution describes a wave travelling in the positive direction of the x-axis. Let compression occur in this wave, i.e.

$$dR / dt = R'(\xi)(U - \xi) / t = -aR'(\xi) / mt > 0$$

This is possible if $R'(\xi) < 0$, or $1/R'(\xi) < 0$. Using (1.16) we obtain

$$\frac{1}{R'(\xi)} = \frac{d\xi}{dR} = \frac{1}{m}\left(\frac{da}{dR} + \frac{a}{R}\right) = \frac{1}{mR} \frac{d(aR)}{dR}$$

This means that a simple wave will be a compression wave only if $d(aR)/dR < 0$. For $d(aR)/dR > 0$ it will be a rarefaction wave.

We examine first the case of elastic shear. Using Formula (1.11) we have

$$\frac{d(a_e R)}{dR} = \frac{1}{2a_e R} \frac{d(a_e R)^2}{dR} = \frac{1}{2a_e R} \Big\{R^2 P''(R) + 2RP'(R) + $$
$$+ \frac{4}{3} n\, [g(R) + Rg'(R)]\Big\} \equiv \frac{1}{2a_e R} A_e(R) \tag{1.17}$$

If $A_e(R) > 0$, then the simple wave will be a rarefaction wave. It is natural to assume that the shear modulus does not decrease with an increase in density, i.e. $g'(R) \geqslant 0$. In order for a simple compression wave to be possible, it is necessary for a region of change of R to exist, in which $P''(R) < 0$. In the opposite case a simple wave will be a rarefaction wave. Thus, for example, if volumetric deformation takes place according to a linearly elastic scheme, then

$$P(R) = C(1 - R_0 / R), \quad C = \text{const}, \quad A_e = \tfrac{4}{3} n (g + Rg') > 0$$

i.e. a simple wave will be a rarefaction wave.

If one passes from R to $\theta = 1 - R_0/R_*$, then the expression for A_e has the form

$$A_e(\theta) = (1-\theta)^2 \left\{P_{\theta\theta}''(\theta) + \frac{4}{3} n \left[\frac{g(\theta)}{1-\theta}\right]_\theta'\right\} \qquad (1.18)$$

From this formula it is seen that (under the condition $g_\theta'(\theta) \geqslant 0$) the necessary condition for the existence of a simple compression wave is $P_{\theta\theta}'' < 0$. We remark here that this condition is by no means sufficient for the existence of such waves. A sufficient condition is the inequality $A_e < 0$.

When shear occurs plastically we have, using Formula (1.12)

$$\frac{d(a_p R)}{dR} = \frac{1}{2a_p R}\frac{d(a_p R)^2}{dR} = \frac{1}{2a_p R}\left[(2RP' + R^2P'') \times \right.$$
$$\left. \times \left(1 \mp \frac{F'}{\sqrt{3F}}\right) \mp \frac{1}{2}(RP')^2 \frac{2FF'' - (F')^2}{2F\sqrt{3F}}\right] \equiv \frac{1}{2a_p R} A_p(R) \qquad (1.19)$$

In the variable θ we have

$$A_p(\theta) = (1-\theta)^2 \left[P_{\theta\theta}''\left(1 \mp \frac{F'}{\sqrt{3F}}\right) \mp (P_\theta')^2 \frac{2FF'' - (F')^2}{2F\sqrt{3F}}\right] \qquad (1.20)$$

If compression occurs in plastic shear, then it is necessary to take the lower sign. Therefore, a simple compression wave with plastic shear can exist only under the condition

$$P_{\theta\theta}''\left(1 + \frac{F'}{\sqrt{3F}}\right) + (P_\theta')^2 \frac{2FF'' - (F')^2}{2F\sqrt{3F}} < 0 \qquad (1.21)$$

while a simple rarefaction wave with plastic shear can exist only under the condition

$$P_{\theta\theta}''\left(1 - \frac{F'}{\sqrt{3F}}\right) - (P_\theta')^2 \frac{2FF'' - (F')^2}{2F\sqrt{3F}} > 0 \qquad (1.22)$$

We note simple cases where waves of one type or another are known to exist or not exist. If $2FF'' - (F')^2 > 0$, then for $P_{\theta\theta}'' > 0$ compression waves do not exist, while for $P_{\theta\theta}'' < 0$ rarefaction waves do not exist. If $2FF'' - (F')^2 < 0$, then for $P_{\theta\theta}'' > 0$ there exists a rarefaction wave, while for $P_{\theta\theta}'' > 0$ there exists a rarefaction wave, while for $P_{\theta\theta}'' < 0$ a compression wave exists. In the intermediate case $2FF'' - (F')^2 \equiv 0$ a more accurate statement can be made: for $P_{\theta\theta}'' > 0$ a rarefaction wave exists and a compression wave does not exist, while for $P_{\theta\theta}'' < 0$ the reverse is the case. It is interesting to note that it is just this intermediate case that occurs in sandy soil for which the function $F(p)$, constructed experimentally [16], has the form

$$F(p) = (kp + b)^2 \tag{1.23}$$

Let us determine the cases in which a simple wave can in fact arise in plastic shear. For this it is necessary to calculate λ by Formula (1.1). By the use of these formulas, and also Formulas (1.5), (1.6), (1.8), (1.16), (1.11), (1.12), we obtain

$$\begin{aligned}\lambda_* = \lambda t &= \pm\frac{3a_p R'(\xi)}{2\sqrt{3F(P)}}\left[\frac{4}{3}n\frac{g(R)}{R} \pm \frac{F'(P)}{\sqrt{3F(P)}}P'(R)\right]\times \\ &\times e\left\{\pm R'(\xi)\left[\frac{4}{3}n\frac{g(R)}{R} \pm \frac{F'(P)}{\sqrt{3F(P)}}P'(R)\right]\right\} \\ &= \pm\frac{3a_p R'(\xi)}{2\sqrt{3F(P)}}(a_e^2 - a_p^2)\, e\,[\pm R'(\xi)(a_e^2 - a_p^2)]\end{aligned} \tag{1.24}$$

In a compression wave $R'(\xi) < 0$ and it is necessary to take the lower sign in Formula (1.24); in a rarefaction wave $R'(\xi) > 0$ and one should take the lower sign in Formula (1.24). In both cases the motion is admissible, i.e. $\lambda > 0$, only for

$$a_e > a_p \tag{1.25}$$

or, what is the same thing, for

$$\frac{4}{3}n\frac{g(R)}{R} \pm \frac{F'(P)}{\sqrt{3F(P)}}P'(R) > 0 \tag{1.26}$$

In rarefaction (upper sign) this inequality is fulfilled everywhere, while in compression (lower sign) it can be fulfilled or not, depending on the properties of the functions $g(R)$, $F(P)$, and $P(R)$. For large values of P this inequality will certainly be violated because $F' < \sqrt{(3F)}$ (see above), $g(R)$ is naturally assumed to be a bounded quantity, and $P'(R)$ may attain a significant magnitude when R approaches the limiting value R_∞ with increasing P. Condition (1.26) and conclusions drawn from it coincide with similar conclusions for the quasistatic problem (see [17], Formula (2.4)). Hence, we see, exactly as in the quasistatic case investigated in [17], that here in the dynamic case in continuous motion the following occurs. Under conditions of rarefaction, plastic shear, once excited, will be retained everywhere under an arbitrary degree of further rarefaction. Under conditions of compression it may disappear and pass over to elastic shear if the further compression is significant. It is of course possible (i.e. such a choice of functions $g(R)$, $F(P)$ and $P(R)$ is possible) that the shear in the compression wave cannot be plastic under any compression, i.e. inequality (1.26) with lower sign is not satisfied for any value of R.

It is convenient to display these results graphically. In Fig. 1 in terms of a plane in the variables $y = -(S + P)$, P the lines OA, OA' show the plastic limit (line (1.8)), and the lines DEB, $D'E'CL$, $D''E''F$ are three out of a set of lines that are determined by the elastic condition (1.9). Each of these lines is determined by the initial point P_{00}, R_{00}, which consequently also specifies the dependence of P on R, and by the initial value $S = S_{00}$. The point C is a point at which condition (1.26) with lower sign is satisfied. The process of deformation of an element of the medium is described in the y, P-plane in the following way. If the initial state is described by the point E (i.e. $S_{00} = -P_{00}$) and rarefaction occurs, then the point describing the state of the element moves along the curve ED (elastic shear) and further along DO (plastic shear). The process is complete when the state of disintegration is reached (point K) for which $R = R_0$, $P = P_0$, which also means $S = S_0$. After this the point jumps to the origin of coordinates: $P = 0$, $S = 0$,

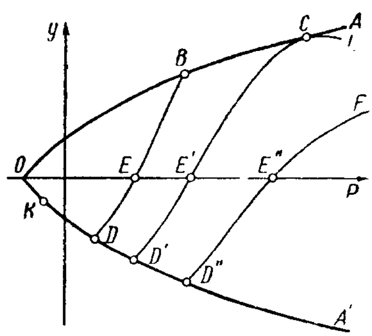

Fig. 1.

which corresponds to stress relief during the disintegration. If a compressive deformation develops from the initial state E, then the point moves along the curve EB, then along BA until the point C is reached, after which it moves along CL. The motion along EB corresponds to an elastic shear deformation, along BC to plastic shear deformation, and along CL to an elastic deformation again. It can be assumed, generally speaking, that upon motion of the point along CL it will again emerge on the curve OA, i.e. plastic shear deformation will begin anew, after which the point will again evolve to a point of the type C, i.e. elastic shear deformation will start again, etc. However, for a significant advance to the right, as has been already mentioned above, the deformation stops and remains elastic for all further times. The coordinates of the points D, B, C and K are easily determined if the dependence $P = P(R)$ is chosen and the functions $F(P)$ and $g(R)$ are given.

2. We turn now to the study of the shock surface in the motions examined. The laws of conservation of mass and momentum lead to the following conditions on the surface of the shock wave:

$$\rho_1 (D - u_1) = \rho_2 (D - u_2) \tag{2.1}$$

$$\sigma_1 + \rho_1 (D - u_1) u_1 = \sigma_2 + \rho_2 (D - u_2) u_2$$

where D is the velocity of the shock wave, the subscript 1 pertains to particles to particles in front of the shock wave, and the subscript 2 pertains to those behind the wave. In the self-similar motion being examined, the velocity of the shock wave is constant, i.e. the parameter $\xi = \xi_*$ is constant on the shock wave. Therefore, using the dimensionless variables (1.5) we obtain from (2.1)

$$\xi_* = \frac{R_2 U_2 - R_1 U_1}{R_2 - R_1}, \qquad S_2 - S_1 = -m^2 R_1 R_2 \frac{(U_2 - U_1)^2}{R_2 - R_1} \tag{2.2}$$

Shock waves which are used to construct the solution of the problem must satisfy necessary conditions of stability, included in the following. The velocity of the wave with respect to the particles in front of the wave should not be smaller than small perturbations in these particle velocities; the velocity of the wave with respect to the particles in back of the wave should not be more than the velocity of small perturbations there [18]. In the problem being examined the quantity $d\sigma/d\rho$ plays the role of a square in the velocity of small perturbations. In the general case jumps that arbitrarily connect two points of possible soil states in the p, ρ-plane are conceivable. We use the following hypothesis to restrict the possible shock waves. We shall assume that points in the p, ρ-plane corresponding to two sides of the shock wave either lie on the curve with the same value of the parameter ρ_*, or, if this is not the case, then at any rate one of the points must certainly be found on the curve $p = p_* = f^{\circ}(\rho_*, \rho_*)$ (see Formula (1.1) as regards notation, and also [15]).

The second of these possibilities can be realized only in jumps of condensation.

These assumptions are justified by the following considerations. In the original notions for a model characterizing volumetric deformation [15], it was assumed that the point in the p, ρ-plane describing the state of the particle can, for $\rho < \rho_*$, only move along the curve $\rho_* =$ const in the process of volumetric deformation, while for $\rho = \rho_*$ it can only move along $p = p_*$. This means that it is possible to go from the point p_1, $\rho_1 < \rho_{*1}$ to the point p_2, $\rho_2 < \rho_{*2} > \rho_{*1}$ only by first moving upward along the line $\rho_* = \rho_{*1}$ and then along the line $p = p_*$ until $\rho_* = \rho_{*2}$ is reached and then downward along the line $\rho_* = \rho_{*2}$. The last stage represents an unloading.

In examining the motion within a thin layer which is changed by the shock surface (i.e. in examining the structure of the shock wave), we shall assume that the process of volumetric deformation occurs only along the described lines. In order for this to be possible it is necessary to assume that there must exist some viscous type forces that have not been taken into account in the equations of the model. The presence of these forces is necessary so that the line in the p, $1/\rho$ plane connecting the points corresponding to the two sides of the shock wave can deviate from the straight-line segment joining these points. However, the deformation process taking place within the thin layer (shock wave) according to such a model will contain a stage of rarefaction if the examined jump as a whole is a jump in condensation. On the other hand, if the jump as a whole is a jump of expansion, then an examination of the jump leads to the conclusion that the jump can only connect points on a line with the same value of ρ_*. Hence, our hypothesis is in essence equivalent (within the framework of the model being examined) to the assumption that within the jumps of condensation the condensation of the particles occurs monotonically.

We turn now to the question of the velocity of the small perturbations which must be used in the study of the stability of shock waves. In the general case of non-self-similar perturbations, the velocity of propagation of the perturbations is also determined by Formulas (1.11) and (1.12), with the proviso, however, that $P'(R)$ should be changed to

$$\frac{dP}{dR} = \left(\frac{\partial P}{\partial R}\right)_{R_*} + \left(\frac{\partial P}{\partial R_*}\right)_R \frac{dR_*}{dR} = \left(\frac{\partial P}{\partial R}\right)_{R_*} + \left(\frac{\partial P}{\partial R_*}\right)_R e(R - R_*)\, e\left(\frac{dR}{dt}\right) \quad (2.3)$$

where the letters next to the parenthesis indicate which argument is held fixed in the differentiation. If $R < R_*$, then the velocities of small perturbations in loading and unloading coincide; however if $R = R_*$, these velocities will be different. If the studied wave is a jump of expansion, then, as has been established by the hypothesis assumed above, the two points of the PR-plane corresponding to the two sides of the jump occur on one and the same line $R_* = \text{const}$. In this case, clearly $R_2 < R_*$ at a point behind the jump, while at a point ahead of the jump either $R_1 = R_*$ or likewise $R_1 < R_*$. If $R_1 = R_*$, then at a point in front of the jump the velocities of small compression and rarefaction will differ. However, in accordance with the requirements for the stability of a jump, the jump velocity must exceed both these velocities. Therefore, one may finally assert that for the stability of rarefaction jumps it is necessary that the jump velocity be smaller than the velocity of small perturbations of the particles in front of the jump and not larger than such velocities in back of the jump, all this under the condition that the velocities of small perturbations be calculated for $R_* = \text{const}$.

In the examination of jumps of condensation, four cases are possible when the densities at points ahead of and behind the jump satisfy: (1) the conditions $R_1 < R_2 < R_{*1}$; (2) the conditions $R_1 < R_2 = R_{*1}$; (3) the conditions $R_1 < R_{*1} < R_2 = R_{*2}$; and (4) the conditions $R_1 = R_{*1} < R_2 = R_{*2}$.

In case (1) it is obvious that the conditions of stability of a jump are formulated in exactly the same way as for the above case of a jump of rarefaction. In case (2) there are again two small perturbation velocities for the particles behind the jump. The velocity of the jump must certainly be no larger than the small perturbation velocity there, calculated under the condition $R_* = \text{const}$; however it may exceed the small perturbation velocity determined by $R = R_*$.

This assumption is justified by the following considerations. If in the neighborhood of the point of intersection of the lines $R_* = \text{const}$ and $R = R_*$, as small a piece as desired of the curve R_* is changed (without a break) so that it smoothly passes over to the curve $R = R_*$, then for stable jumps the condition $R_2 < R_{*1}$ will certainly be fulfilled, R_2 differing from R_{*1} as little as desired. Thereby case (2) is reduced to case (1). Passing now to the limit (setting the length of the smoothed section of the curve $R_* = \text{const}$ to zero), we obtain case (2), and the conditions of stability of the shock wave remain identical to the conditions for case (1). In case (3), for a particle in back of the wave, the small perturbation velocities have two values. The condition of stability requires that the wave velocity does not exceed both values, and since the velocity of loading perturbations (i.e. calculated under the condition $R = R_*$) is the smaller of the two values, it is necessary for stability that just this velocity be not exceeded by the wave velocity. The condition at a particle in front of the jump is the same as that in cases (1) and (2). Finally, in case (4), the stability condition at a particle in back of the jump coincides with the stability condition for case (3), while for a particle in front of the jump the situation is analogous to case (2). Here we likewise reduce case (4) to case (3) by smoothing the transition from the curve $R_* = \text{const}$ to the curve $R = R_*$, and as a result of a limiting transition we establish that for stability it is sufficient for the wave velocity with respect to a particle in front of the wave to be not smaller than the small perturbation velocities there, determined under the condition $R = R_*$.

It is easy to unify all of these cases into the following simple rule. If a shock wave of compression (rarefaction) produces a condensation (rarefaction) of the medium from the density R_1 to the density R_2, then for the shock wave to be stable its velocity relative to the particles should not be smaller than the small perturbation velocities of particles in front of the wave and not larger than the small perturbation velocities of particles behind the wave, the perturbation velocities being calculated according to a dependence of P on R that connects the points P_1, R_1 and P_2, R_2 in a unique possible way. This rule greatly simplifies the investigation of shock waves because it allows one to bypass the dependence of P on R or R_* and in every case deals with a single relation $P = P(R)$ connecting the points describing the state of the particles in front of and behind the wave.

The formulated rule only gives us stability conditions for the shock wave. For this or that wave satisfying this rule to actually exist, it is necessary, generally speaking, to satisfy additional sufficiency conditions.

For certain cases we next determine conditions which allow one to establish the known stability or instability of waves. Let a compression from R_1 to R_2 occur in a shock wave. The stability conditions read

$$|\xi_* - U_1| = R_2 \left|\frac{U_2 - U_1}{R_2 - R_1}\right| \geqslant \frac{1}{m} a(R_1) \qquad (2.4)$$

$$|\xi_* - U_2| = R_1 \left|\frac{U_2 - U_1}{R_2 - R_1}\right| \leqslant \frac{1}{m} a(R_2) \qquad (2.5)$$

where, by virtue of what has been said above, $a(R)$ may be determined by Formulas (1.11), (1.12). If $d(aR)^2/dR \geqslant 0$ in the interval $R_1 \leqslant R \leqslant R_2$, then as a result of the second of Formulas (2.2) and Formulas (1.11), (1.12) we have

$$m^2 R_1 R_2 \frac{(U_2 - U_1)^2}{R_2 - R_1} = S_1 - S_2 = \int_{R_1}^{R_2} a^2(R)\, dR$$

$$= \int_{R_1}^{R_2} (aR)^2 \frac{dR}{R^2} \geqslant a^2(R_1) R_1^2 \int_{R_1}^{R_2} \frac{dR}{R^2} = \frac{R_2 - R_1}{R_1 R_2} R_1^2 a^2(R_1)$$

Hence it follows that condition (2.4) is satisfied. It can be analogously established that condition (2.5) is satisfied as well. However if $d(aR)^2/dR \leqslant 0$ then in an exactly analogous way it can be shown that the stability conditions are not satisfied, i.e. that the compressive wave is unstable in this case. In rarefaction, i.e. for $R_2 < R_1$ we may analogously establish that for $d(aR)^2/dR \leqslant 0$ in the interval $R_2 \leqslant R \leqslant R_1$ a shock wave of expansion is stable, while for $d(aR)^2/dR \geqslant 0$ $(R_2 \leqslant R \leqslant R_1)$ it is unstable. Comparing the results obtained here with the results of Section 1, we see that if $d(aR)/dR \geqslant 0$, then the existence of continuous rarefaction waves and stable compressive shock wave is possible. On the other hand if $d(aR)/dR \leqslant 0$, the existence of continuous compressive waves and stable rarefaction shock waves is possible.

The case of the quantity $d(ar)/dR$ changing sign in the region of R being studied is complicated and requires a more detailed investigation.

3. We turn now to the construction of the solution of the piston problem. We examine first the case where the piston, moving to the left out of the medium, produces a rarefaction. Here it is required to construct a solution of Equations (1.10) that satisfy the following conditions: for $\xi = -1, U = -1$ and for $\xi \to \infty$, $U = 0$, $R = R_{00}$. We shall assume that $d(a_e R)/dR \geqslant 0$ and $d(a_p R)/dR \geqslant 0$. Under these conditions, as has been established above, there can exist continuous rarefaction waves and stable jumps of condensation, but there cannot exist stable jumps of rarefaction. For simplicity we assume that in the initial state $S_{00} = -P_{00}$. Let the velocity of the piston be extremely small. Then the parameter m is small and the whole motion will be small. The motion cannot be contiguous to a region of the initial rest state of the compressive shock wave because in this case a progressive flow or a rarefaction wave could follow in back of the shock. However, neither of these is possible because in view of the stability of the shock wave its velocity along the particles in back of it is smaller than the sound velocity there, and therefore in the case of a progressive flow in back of the jump the second boundary of the region of the progressive flow, moving at the sound velocity, and hence faster than the jump, would not be able to remain away from the jump. In the case of a rarefaction wave in back of the jump, the same reason would cause the wave to overtake the jump and weaken it. This would be contrary to the constancy of the magnitude of the jump as given by the self-similar solution of the problem. Hence, the motion in the neighborhood of the region of the initial rest state will certainly be a rarefaction wave, and since $S_{00} = -P_{00}$ in the initial state, the shear in this wave will occur elastically. If the velocity of the piston is sufficiently small, then the shear will be everywhere elastic in the region of motion, and the solution will consist of the indicated expansion wave and a region of progressive motion bordering on the wave of rarefaction and extending to the piston. Compressive waves in interior portions of the region of motion also cannot exist (for the same reasons as above). The solution has the form

$$
\begin{gathered}
U = 0, \qquad R = R_{00} \qquad (\xi_0 \leqslant \xi \leqslant +\infty) \\
U = -\frac{1}{m} \int_R^{R_{00}} \frac{a_e(R)\,dR}{R}, \qquad U = \xi - \frac{1}{m} a_e(R) \qquad (\xi_1 \leqslant \xi \leqslant \xi_0) \qquad (3.1) \\
U = -1, \qquad R = R_{\min} \qquad (-1 \leqslant \xi \leqslant \xi_1)
\end{gathered}
$$

The constant parameters ξ_0, ξ_1 and $R_{\min}$ are determined from the conditions of continuity of the solution on the boundaries of the expansion wave and the regions of the progressive flow and the condition on the piston

$$\xi_0 = \frac{1}{m} a_c(R_{00}), \qquad \int\limits_{R_{\min}}^{R_{00}} \frac{a_e(R)}{R} dR = m, \ \xi_1 = -1 + \frac{1}{m} a_e(R_{\min}) \tag{3.2}$$

From these formulas it is seen that for small velocities of the piston, i.e. for small m, $R_{\min}$ will be close to R_{00} and the motion will be small and concentrated in a thin layer between the near surfaces $\xi = \xi_0$ and $\xi = \xi_1$. With an increase in the velocity of the piston the rarefaction will grow; $R_{\min}$ will move away from R_{00}, and the width of the rarefaction wave will increase. Subsequently, either the elastic limit in shear will be reached and a solution with plastic shear will have to be found, or a state will be reached under elastic shear where the stress approaches zero in the region of the progressive wave abutting the piston. If the particles of the medium that are immediately contiguous to the piston are only in free contact with it, then a further increase in the velocity of the piston will not correspond to motion of the medium. The piston will tear away from the medium and a vacuum will be formed between the piston and the surface of the soil. Likewise, if the particles directly next to the piston are rigidly attached to the piston ("glued on"), then an increased velocity of the piston will correspond to that of the medium as long as the pressure P in the region of the progressive flow does not attain the minimum possible value $P_0 < 0$, below which the medium will not sustain tensile stresses. For velocities of the piston exceeding this critical value, there will occur at time $t = 0$ a discontinuity in the medium at the cross-section directly adjacent to the piston, so that the motion will coinside with the motion which is excited by the velocity of the piston when $S = 0$ at the piston. The feasibility of constructing a solution with a jump which changes the particles from a state of limiting tension $P = P_0 < 0$, $R = R_0$, $S = S_0 > 0$ to the disintegration state $P = 0$, $S = 0$ does not arise because, as is seen from the second of Formulas (2.2), condensation would occur on such a surface of discontinuity, which contradicts the assumption that disintegration also occurs behind the surface $P = S = 0$.

The velocity of the piston for which $S = 0$ on the piston is determined from the condition

$$m = \int\limits_{R_*}^{R_{00}} \frac{a_e(R)}{R} dR \tag{3.3}$$

where the density R_* is, by virtue of (1.9), determined from the condition

$$P(R_*) - \frac{4}{3} n \int\limits_{R_*}^{R_{00}} \frac{g(R)}{R} dR = 0 \tag{3.4}$$

Likewise, in the case of the glued piston the velocity at which the piston attains the limiting tension is determined from the condition

$$m = \int_{R_0}^{R_{00}} \frac{a_e(R)}{R} dR \tag{3.5}$$

where R_0 is the minimum density possible without disintegration [15]. Clearly $R_0 < R_*$. Of course, all of this will take place if the shear remains elastic for all time during a change of density from R_{00} to R_* and R_0, i.e. the point in the y, P-plane remains above the line OA' (Fig. 1). However, if there exists a value of the density $R_D > R_*$, at the attainment of which the point emerges on the line OA', then for larger rarefactions, i.e. for larger velocities of the piston, it is necessary to construct a solution which takes into account the formation of a region of plastic shear. We show at the outset that the transition into the plastic shear region cannot occur with a jump. This jump cannot be a jump of condensation because it would leave the particle in the elastic region, and for this case it has been shown above that a jump of condensation cannot exist in the solution. Hence, the jump, if it exists, must be a jump of rarefaction. It cannot be entirely in the elastic region or entirely in the plastic region because it would then be unstable. This means that it can transfer a particle from the elastic to the plastic region. Using (2.2) we have, taking into account that $a_p < a_e$

$$m^2 R_1 R_2 \frac{(U_2 - U_1)^2}{R_1 - R_2} = S_2 - S_1 = \int_{R_2}^{R_D} a_p^2(R)\, dR + \int_{R_D}^{R_1} a_e^2(R)\, dR <$$

$$< \int_{R_2}^{R_1} a_e^2(R)\, dR = \int_{R_2}^{R_1} (a_e R)^2 \frac{dR}{R^2} < a_e^2(R_1) R_1^2 \frac{R_1 - R_2}{R_1 R_2}$$

Hence, we obtain the condition of the instability of a discontinuity

$$|\xi_* - U_1| = R_2 \left| \frac{U_2 - U_1}{R_2 - R_1} \right| < \frac{1}{m} a_e(R_1)$$

Thus, the transition from the region of elastic shear to the region of plastic shear occurs continuously. However, the rarefaction wave which certainly occurs in the plastic region cannot be adjacent to the rarefaction wave in the elastic region, since the velocity of sound undergoes a discontinuity in the transition from the elastic to the plastic region. This means that between these two waves of rarefaction there must exist a region of progressive motion. In conclusion, the solution has the form

$$
\begin{array}{lll}
U = 0, \quad R = R_{00} & & (\xi_0 \leqslant \xi \leqslant +\infty) \\
U = -\dfrac{1}{m} \displaystyle\int\limits_R^{R_{00}} \dfrac{a_e(R)}{R} dR, & U = \xi - \dfrac{1}{m} a_e(R) & (\xi_{De} \leqslant \xi \leqslant \xi_0) \\
U = U_D, \quad R = R_D & & (\xi_{Dp} \leqslant \xi \leqslant \xi_{De}) \\
U = -1 + \dfrac{1}{m} \displaystyle\int\limits_{R_{\min}}^{R} \dfrac{a_p(R)}{R} dR, & U = \xi - \dfrac{1}{m} a_p(R) & (\xi_1 \leqslant \xi \leqslant \xi_{Dp}) \\
U = -1, \quad R = R_{\min} & & (-1 \leqslant \xi \leqslant \xi_1)
\end{array}
\qquad (3.6)
$$

The value of the density R_D is determined by the condition of exit to point D (Fig. 1), while the constants ξ_0, ξ_{De}, ξ_{Dp}, ξ_1, U_D, $R_{\min}$ are determined from obvious continuity conditions, giving the formulas

$$
U_D = -\frac{1}{m} \int\limits_{R_D}^{R_{00}} \frac{a_e(R)}{R} dR, \qquad \int\limits_{R_{\min}}^{R_D} \frac{a_p(R)}{R} dR + \int\limits_{R_D}^{R_{00}} \frac{a_e(R)}{R} dR = m
$$

$$
\xi_0 = \frac{1}{m} a_e(R_{00}), \qquad \xi_1 = -1 + \frac{1}{m} a_p(R_{\min}) \qquad (3.7)
$$

$$
\xi_{De} = U_D + \frac{1}{m} a_e(R_D), \qquad \xi_{Dp} = U_D + \frac{1}{m} a_p(R_D)
$$

Here, with an increase of the piston velocity V, $R_{\min}$ will likewise decrease, reaching first the value R_{**}, for which $S = 0$, and then R_0 (for the glued piston). The formulas for R_{**} and the corresponding velocity will be

$$
P(R_{**}) - \tfrac{2}{3} \sqrt{3F\,[P(R_{**})]} = 0
$$

$$
m = \int\limits_{R_{**}}^{R_D} \frac{a_p(R)}{R} dR + \int\limits_{R_D}^{R_{00}} \frac{a_e(R)}{R} dR \qquad (3.8)
$$

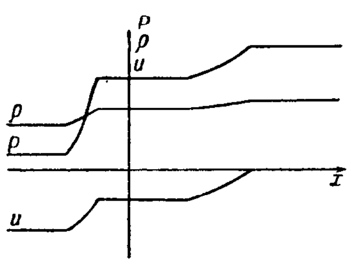

Fig. 2.

Figure 2 shows an example of the form of the diagram of the parameters of motion for the solution (3.6). The solution constructed, containing a portion with plastic shear, has a remarkable property. It contains two centered rarefaction waves, separated by a region of progressive motion. This type of motion is not possible in a gas and in general in an ideal fluid having a sound velocity which is a continuous function of density. A similar motion occurs in magnetohydrodynamics where the medium is characterized by two sound velocities. These also make possible the separate propagation of two centered rarefaction waves in the self-similar problem.

We now investigate the case of motion of the piston to the left, but $d(a_e R)/dR \leqslant 0$ and $d(a_p R)/dR \leqslant 0$. We again construct the solution, starting with the smallest values of the piston velocity and then increasing these velocities. For small velocities the motion will be small, and therefore the shear will be elastic throughout the region of motion. By virtue of the assumed conditions, continuous waves of rarefaction and stable compressive shock waves will not be possible. By means of considerations completely analogous to those introduced above, it can be shown that the solution of the problem cannot contain continuous compressive waves. Therefore, we finally obtain a solution consisting of a progressive flow bounded from the side of the region of the initial unperturbed state by a rarefaction jump. It is given by the formula

$$\begin{aligned} U &= 0, \qquad R = R_{00} \qquad (\xi_0 \leqslant \xi \leqslant +\infty) \\ U &= -1, \qquad R = R_{\min} \qquad (-1 \leqslant \xi \leqslant \xi_0) \end{aligned} \tag{3.9}$$

Here the constants $R_{\min}$ and ξ_0 are determined from conditions on the surface of the jump by the formulas

$$\int_{R_{\min}}^{R_{00}} a^2{}_e(R)\, dR = m^2 \frac{R_{00} R_{\min}}{R_{00} - R_{\min}}, \qquad \xi_0 = \frac{R_{\min}}{R_{00} - R_{\min}} \tag{3.10}$$

These formulas solve the problem for $R_{\min} \geqslant R_D$, where R_D is the density for which the elastic limit in shear is first attained. This takes place when the velocity of the piston attains the value V_D determined from the formula

$$\int_{R_D}^{R_{00}} a_e{}^2(R)\, dR = m^2 \frac{R_{00} R_D}{R_{00} - R_D} \tag{3.11}$$

For larger values of the velocities the solution will contain regions of plastic shear.

To construct the solution for such velocities, we examine the relationship between the sound velocity under plastic shear for $R = R_D$ and the velocity of a jump along the particles in back of it for $V = V_D$. For these quantities we have the formulas

$$\frac{1}{m} a_p(R_D) = \frac{1}{mR_D} \left(\frac{dS_p}{d\,\dfrac{1}{R}}\right)_D^{1/2}, \qquad W_D \equiv |\xi_D - U_{2D}| = \frac{1}{mR_D} \sqrt{\frac{|S_D - S_{00}|}{|R_D{}^{-1} - R_{00}{}^{-1}|}} \tag{3.12}$$

If the velocity of the piston does not significantly exceed V_D, and if a solution of a single wave is constructed as previously, the density in back of the wave R_2 will differ little from R_D, $a_p(R_2)$ from $a_p(R_D)$ and W_2 from W_D. Therefore, if $a_p(R_D)/m < W_D$ then $a_p(R_2)/m < W_2$ as well for these velocities V. This means that the shock wave in such a solution will be unstable. The only possible solution for these values of the piston velocity will be a solution consisting of two rarefaction shock waves moving one after the other. The solution is given by the formula

$$\begin{aligned} &U = 0, \quad R = R_{00} \quad (\xi_D \leqslant \xi \leqslant +\infty) \\ &U = U_D, \quad R = R_D \quad (\xi_1 \leqslant \xi \leqslant \xi_D) \\ &U = -1, \quad R = R_{\min} \quad (-1 \leqslant \xi \leqslant \xi_1) \end{aligned} \tag{3.13}$$

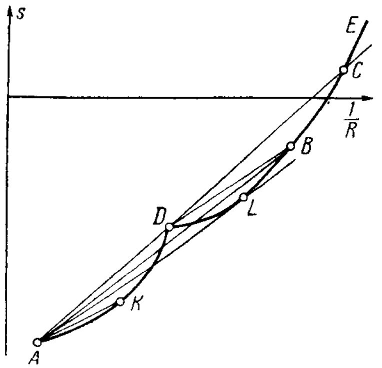

Fig. 3.

Here the constants U_D, ξ_D, $R_{\min}$, ξ_1 are found from corresponding conditions on the shock waves from the formulas

$$\xi_D = -\frac{R_D U_D}{R_{00} - R_D}, \qquad U_D = -\frac{1}{m}\sqrt{\frac{R_{00} - R_D}{R_{00} R_D}}\left(\int_{R_D}^{R_{00}} a_e(R)\,dR\right)^{\frac{1}{2}}$$

$$\int_{R_{\min}}^{R_D} a_p^2(R)\,dR = m^2 \frac{(1 + U_D)^2}{R_D - R_{\min}} R_D R_{\min}, \qquad \xi_1 = \frac{R_D U_D + R_{\min}}{R_D - R_{\min}} \tag{3.14}$$

In Fig. 3 the curve AD shows the connection between S and $1/R$ for elastic shear and the curve $DBCE$ for plastic shear. Point A is the initial state, point D the state behind the first shock wave (transition state), point B the state in back of the second shock wave, point K the state behind the shock wave when $V < V_D$. Under an increase in the piston velocity V the point describing the state in back of the shock moves from the point A to the point D along AKD; after point D is reached, the velocity of the shock wave and the state of the medium in back of it does not change with a further increase in V. Rather, a second wave is excited with a point that moves from point D along DBE. If the point C exists (the intersection of the straight line AD and the curve DE), then when it is reached the second wave catches up with the first wave, and for larger values of the velocity V the motion again occurs with a single wave. The value of the velocity at which this occurs is given by the relation

$$\xi_1(m_c) = \xi_D(m_c) \tag{3.15}$$

For $m > m_c$ the solution is determined by the formulas

$$
\begin{gathered}
U = 0, \qquad R = R_{00} \qquad (\xi_0 \leqslant \xi \leqslant +\infty) \\
U = -1, \qquad R = R_{\min} \qquad (-1 \leqslant \xi \leqslant \xi_0) \\
\int_{R_{\min}}^{R_D} a_p^2(R)\, dR + \int_{R_D}^{R_{00}} a_e^2(R)\, dR = m^2 \frac{R_{00} R_{\min}}{R_{00} - R_{\min}} \\
\xi_0 = \frac{R_{\min}}{R_{00} - R_{\min}}
\end{gathered}
\tag{3.16}
$$

Here, for a certain piston velocity, the state for which $S = 0$ on the piston will be likewise attained, and there will also exist a value of the velocity for which the state of limiting tension will be reached in the neighborhood of the piston. These states can be reached up to the attainment of states D and C and after state C (Fig. 3). In each of these cases it is a simple matter to write down formulas which determine the values of the velocities for which these limiting states are attained. The formulas are similar to (3.3) to (3.5), and (3.8).

If the functions $a_e(R)$ and $a_p(R)$ are such that $a_p(R_D)/m \geqslant W_D$, then there will be no second wave regardless of the value of V. It should be mentioned that for the points on Fig. 3 which are located between C and L (L is the point at which a straight line emanating from point A is tangent to the curve DE) it is possible, in addition to the above solution, to construct a second solution in which there occurs only one shock wave, connecting the points A and B directly. This wave is clearly stable. However this solution must be discarded because the solution, under the condition that the stability conditions are fulfilled at A and B, does not depend on the form of the curve ADB in the interval between A and B. Hence, in particular, the curve in this interval can be arranged so that the necessary thermodynamical energy inequality [19,20] is not satisfied on the shock wave, since the deformation work entering into this inequality depends on the form of the curve ADB.

When the quantities $d(a_e R)/dR$ and $d(a_p R)/dR$ change sign in the range of R of interest, the situation can be studied in an analogous fashion. For the piston problem in these cases there will be regions with rarefaction waves and a progressive flow, and likewise rarefaction shock waves.

We turn our attention now to a curious property of the solution that has been constructed – it can contain two shock waves, propagating at different velocities from one another. This situation is likewise related to the discontinuous character of the sound velocity as a function of the density and is a consequence of the transition from elastic to plastic shear deformation. In ordinary ideal fluids this phenomenon cannot take place. Such a phenomenon is met in magnetohydrodynamics where it is likewise associated with the presence of two sound velocities.

We turn finally to the case of the piston moving to the right and producing compression. The considerations introduced above for the motion producing rarefaction carry over completely. A difference will arise in the following two circumstances. In rarefaction motion the sound velocity underwent a discontinuity only in the transition from elastic to plastic shear. With this, as established above, was associated the possibility of two shock or continuous rarefaction waves existing. In compressive motions, the sound velocity will undergo a discontinuity at another point in addition to the point of transition from elastic to plastic shear. This is the point in the P, R-plane at which there occurs a transition from compression along the curve $R_* =$ const to the curve $R = R_*$. At this point dP/dR undergoes a discontinuity, as is seen from Formula (2.3), and this means that the sound velocity a suffers a discontinuity as well ((1.11), (1.12)).

Further, if $d(\dot{a}R)/dR \leqslant 0$ were true for large values of P, then as can be verified from an analysis of the equations expressing the solution of the problem, the solution can be constructed only for a range of values of the piston velocity V which is bounded from above. But the piston velocity is an assigned external quantity. In order for the problem, whose solution must exist on physical grounds, to be solvable for arbitrary V, it is necessary to assume that $d(aR)/dR > 0$ for significant densities. This condition will be known to be satisfied if, for example, the density of the medium R remains bounded for an unbounded increase in the pressure P.

For an increase in the density R, both for a transition through the elastic limit in shear and for a transition through a break in the P, R diagram, the sound velocity, changing by a jump, decreases. This will lead to the situation that when $d(aR)/dR < 0$ there will be, corresponding to values of the density at which the sound velocity undergoes a discontinuity, regions of progressive motion enclosed between two continuous compressive waves. When $d(aR)/dR > 0$ this will lead to the increase in the number of compressive shock waves by one, for a transition through each value of density at which the sound velocity undergoes a discontinuity. The considerations which establish these facts are almost a literal repetition of the analogous considerations which were introduced above in the case of rarefaction. Hence, even when the condition $d(aR)/dR > 0$ is satisfied throughout the region of continuous sound velocity $a(R)$, there can exist three compressive shock waves travelling one after the other at different velocities for sufficiently large values of the piston velocity.

Further, if it is possible for the function $d(aR)/dR$ to change sign several times, then the number of shock waves can be still larger.

If for values of R close to R_{00}, $d(aR)/dR < 0$, while for larger R, $d(aR)/dR > 0$, then for small piston velocities the motion will be a continuous compressive wave. With a growth in the piston velocity, a jump will emerge in back of the continuous wave. The jump will move into the main part of the continuous wave with increasing piston velocity, absorbing it, so that at a certain value of the piston velocity the continuous compressive wave will dissipate and a single jump will remain.

An experiment [16] shows that exactly this case occurs in reality. Therefore, in the solution of specific problems more attention should be paid to it.

It should be emphasized that inasmuch as for $P \to \infty$ it is certainly true that $d(aR)/dR > 0$ (as was established above), in all cases, starting with some value of the piston velocity, the piston motion for all large values of the piston velocity will represent a progressive flow, bounded uniquely from the direction of the unperturbed medium in the solution of the shock wave problem.

We remark finally that the transition from plastic shear to elastic shear again cannot be accompanied by any sort of qualitative change in the solution of the problem, since from the relations (1.25), (1.26) it follows that this transition occurs with the sound velocity remaining continuous.

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